TRINITY COLLEGE

Year 11 Mathematics Specialist
Test 6 2016

Calculator Free
Mathematical induction and complex numbers

STUDENT'S NAME ______________________

DATE: **TIME:** 50 minutes **MARKS:** 51

INSTRUCTIONS:
Standard Items: Pens, pencils, ruler, eraser.
Special Items: Formula sheet

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (5 marks)

If $(a + bi)^2 = 3 + 4i$, where a and b are real numbers, determine the values of a and b.

$a^2 + 2abi - b^2 = 3 + 4i \Rightarrow a^2 - b^2 = 3$

$2ab = 4 \Rightarrow b = \frac{2}{a}$

$\therefore a^2 - \frac{4}{a^2} = 3$

i.e. $a^4 - 3a^2 - 4 = 0$

$(\cancel{a^2+1})(a^2 - 4) = 0$

$a = \pm 2 \therefore b = \pm 1$

2. (3 marks)

Determine the quadratic equation whose roots are $1 + 5i$ and $1 - 5i$.

$$\left[x - (1+5i)\right]\left[x - (1-5i)\right] = 0$$

$$\Rightarrow x^2 + (1+25) - x(1+5i) - x(1-5i) = 0$$

$$\Rightarrow \underline{x^2 - 2x + 26 = 0}$$

OR

$$x = 1 \pm 5i$$

$$= 1 \pm \sqrt{-25}$$

$$x - 1 = \pm\sqrt{-25}$$

$$(x-1)^2 = -25$$

$$(x-1)^2 + 25 = 0$$

ie $\underline{x^2 - 2x + 26 = 0}$

3. (4 marks)

One root of the equation $z^2 + az + b = 0$, where a and b are real constants, is $2 + 3i$. Determine the values of a and b.

$$(2+3i)^2 + a(2+3i) + b = 0$$

$$\Rightarrow 4 + 12i - 9 + 2a + 3ai + b = 0$$

$$\Rightarrow -5 + 2a + b + i(3a+12) = 0 \Rightarrow 3a + 12 = 0 \Rightarrow \underline{a = -4}$$

$$\& -5 + 2a + b = 0 \Rightarrow \underline{b = 13}$$

4. (6 marks)

The complex number z satisfies $\dfrac{z}{z+2} = 2 - i$. Determine the real and imaginary parts of z. (Hint: let $z = a + bi$).

$$\Rightarrow \quad z = (2-i)(z+2)$$

$$= 2z + 4 - iz - 2i \qquad \text{—— } (1)$$

$$\text{i.e.} \quad (a+bi) = (2-i)(a+bi) + 4 - 2i$$

$$= (2a+4+b) + i(2b-a-2)$$

$$\Rightarrow \quad 2a+4+b = a \quad \& \quad 2b-a-2 = b$$

$$\Rightarrow \quad a+b = -4 \quad \& \quad a-b = -2$$

$$\therefore 2a = -6$$

$$\underline{a = -3}$$

$$\therefore \quad \underline{b = -1}$$

OR from (1)

$$-z + iz = 4 - 2i$$

$$z(-1+i) = 4 - 2i$$

$$z = \frac{4-2i}{-1+i} \cdot \frac{-1-i}{-1-i}$$

$$= \frac{(-4-2) + i(2-4)}{2}$$

$$= \underline{-3 - i} \quad \Rightarrow a = -3,\ \underline{b = -1}$$

$$\frac{a+bi}{(a+2)+bi} \cdot \frac{(a+2)-bi}{(a+2)-bi}$$

$$= \frac{a(a+2)+b^2+i(2b)}{(a+2)^2+b^2}$$

$$= \frac{a^2+2a+b^2}{(a+2)^2+b^2} + \frac{i\,2b}{(a+2)^2+b^2} = 2-i$$

$$\Rightarrow \frac{a^2+2a+b^2}{(a+2)^2+b^2} = 2 \qquad \& \quad \frac{2b}{(a+2)^2+b^2} = -1$$

$$\Rightarrow \cancel{a^2}+\cancel{b^2}+2a = \cancel{2}a^2+8a+8+\cancel{2}b^2$$

$$0 = a^2+b^2+6a+8$$

$$a^2+b^2 = -6a-8$$

$$2b = -a^2-4a-4-b^2$$

$$0 = a^2+b^2+4a+2b+4$$

$$\rightarrow 0 = -6a \cancel{-8} +4a+2b+4$$

$$= -2a-4+2b$$

$$b = a+2$$

$$\therefore a^2+(a+2)^2 = -6a-8$$

$$2a^2+4a+4+6a+8 = 0$$

$$2a^2+10a+12 = 0$$

$$a^2+5a+6 = 0$$

$$(a+3)(a+2) = 0 \Rightarrow a = -3 \text{ or } \cancel{-2} \;\; \neq 0$$

$$b = -1 \text{ or } \cancel{0}$$

$$\therefore \boxed{\begin{array}{l} a = -3 \\ b = -1 \end{array}}$$

5. (9 marks)

Simplify the following complex expressions leaving your answer in the form $a + bi$

(a) $2 - i - (-3 + 2i)$ [1]

$= 5 - 3i$

(b) $(3 - 2i)(-2 + 5i)$ [2]

$= (-6 + 10) + i(15 + 4)$

$= 4 + 19i$

(c) $\dfrac{-3 - i}{2 + 3i} \cdot \dfrac{2 - 3i}{2 - 3i}$ [4]

$= \dfrac{(-6 - 3) + i(9 - 2)}{13}$

$= -\dfrac{9}{13} + \dfrac{7}{13}i$

(d) $\dfrac{i}{-i^3}$ [2]

$= \dfrac{i}{i} = 1$

6. (5 marks)

Using the principle of mathematical induction prove

$$2^0 + 2^1 + 2^2 + \ldots + \ldots 2^n = 2^{n+1} - 1 \quad \text{for } n \geq 0 \qquad (1)$$

$n=0$, LHS $= 2^0 = 1$, RHS $= 2^1 - 1 = 1$ $\therefore$ true for $n=0$

assume true for $n=k$

ie $2^0 + 2^1 + 2^2 + \ldots + 2^k = 2^{k+1} - 1$

try for $n=k+1$

LHS $= 2^0 + 2^1 + 2^2 + \ldots + 2^k + 2^{k+1}$

$= 2^{k+1} - 1 + 2^{k+1}$ (by assumption)

$= 2 . 2^{k+1} - 1$

$= 2^{k+2} - 1$ which is (1) with $n = k+1$

$\therefore$ by Pr of MI, (1) is true $\forall n \geqslant 0$

7. (6 marks)

Using the principle of mathematical induction prove that $9^n - 2^n$ is divisible by seven for $n \in \mathbb{Z}^+$

$n=1 \Rightarrow 9^1 - 2^1 = 7 \div 7$ $\therefore$ true for $n=1$

assume true for $n=k$

i.e. $9^k - 2^k = 7q, \; q \in \mathbb{Z}$

$n=k+1$ $\quad 9^{k+1} - 2^{k+1}$

$= 9.9^k - 2.2^k$

$= (7+2).9^k - 2.2^k$

$= 7.9^k + 2.9^k - 2.2^k$

$= 7.9^k + 2(7q)$ by assumption

$= 7[9^k + 2q]$ $\div$ b 7

$\therefore 9^n - 2^n$ is $\div 7$ for $n \in \mathbb{Z}^+$ by Pr of MI

8. (7 marks)

Prove $\begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix}^n = \begin{bmatrix} -3n+1 & 9n \\ -n & 3n+1 \end{bmatrix}$ for $n \geq 1$ using mathematical induction ①

$n=1$, LHS $= \begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix}$, RHS $= \begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix}$ $\therefore$ true for $n=1$

assume true for $n=k$

ie $\begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix}^k = \begin{bmatrix} -3k+1 & 9k \\ -k & 3k+1 \end{bmatrix}$

try for $n=k+1$

$$\text{LHS} = \begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix}^{k+1}$$

$$= \begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix}^k \begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -3k+1 & 9k \\ -k & 3k+1 \end{bmatrix} \begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix} \quad \text{by assumption}$$

$$= \begin{bmatrix} 6k-2-9k & -27k+9+36k \\ 2k-3k-1 & -9k+12k+4 \end{bmatrix}$$

$$= \begin{bmatrix} -3k-2 & 9k+9 \\ -k-1 & 3k+4 \end{bmatrix}$$

$$= \begin{bmatrix} -3(k+1)+1 & 9(k+1) \\ -(k+1) & 3(k+1)+1 \end{bmatrix}$$ $\therefore$ ① is true for $n=k+1$

$\Rightarrow \begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} -3n+1 & 9n \\ -n & 3n+1 \end{bmatrix}$

is true $\forall n \geq 1$

by Pr of MI

9. (6 marks)

$2(k+1)-1$
$2k+1$

Prove that $\cos x + \cos 3x + \cos 5x + \ldots + \cos[(2n-1)x] = \dfrac{\sin 2nx}{2\sin x}$ for $n \in \mathbb{Z}^+$ ①

$n=1$ LHS $= \cos x$, RHS $= \dfrac{\sin 2x}{2\sin x} = \dfrac{\cancel{2\sin x}\cos x}{\cancel{2\sin x}}$

$= \cos x$ $\therefore$ true for $n=1$

assume true for $n=k$

ie $\cos x + \cos 3x + \cos 5x + \ldots + \cos[(2k-1)x] = \dfrac{\sin 2kx}{2\sin x}$

try $n=k+1$

LHS $= \cos x + \cos 3x + \cos 5x + \ldots \cos[(2k-1)x] + \cos[(2k+1)x]$

$= \dfrac{\sin 2kx}{2\sin x} + \cos[(2k+1)x]$ by assumption

$= \dfrac{\sin 2kx + 2\sin x \cdot \cos[(2k+1)x]}{2\sin x}$

$x + 2kx + x$
$= 2x + 2kx$

$\cancel{x} - 2kx - \cancel{x}$

$= \dfrac{\sin 2kx + \sin[x+(2k+1)x] + \sin[x-(2k+1)x]}{2\sin x}$

$= \dfrac{\sin 2kx + \sin[2(k+1)x] + \sin[-2kx]}{2\sin x}$ $= -\sin 2kx$

$= \dfrac{\sin[2(k+1)x]}{2\sin x}$ which is ① with $n=k+1$

$\therefore$ ① is true for $n \in \mathbb{Z}^+$

by Pr of MI